$$(m)^n+(m-1)^n+(m-2)^n+....2^n+1^n=m'(2m+1)....(48),$$

where $2m+1$ is prime, and where n is any even number less than $2m$.

Thus, 17 will exactly divide

$$8^{2x}+7^{2x}+6^{2x}+5^{2x}+4^{2x}+3^{2x}+2^{2x}+1,$$

when $x=1, 2, 3, 4, 5, 6,$ or 7.

The converse of either of the preceding properties is not always true; we will now deduce some properties which belong exclusively to prime numbers.

According to Wilson's theorem, m' being an integer,

$$1+n!=m'(n+1)....(49)$$

only when $n+1$ is a prime number.

When $n+1$ is a prime number, (42) may be written

$$n!=(c+n)^n+(c+n-1)^n+(c+n-2)^n+....c^n-m'(n+1)....(50).$$

Adding 1 to both sides and we readily obtain

$$(c+n)^n+(c+n-1)^n+(c+n-2)^n+....c^n+1=m'(n+1)....(51),$$

which is true only when $n+1$ is a prime number.

Making $c=0$, and we have

$$n^n+(n-1)^n+(n-2)^n+....1+1=m'(n+1)....(52).$$

That is, S_n+1 is exactly divisible by $n+1$, when the latter is a prime number, and only when it is prime, where $S_n=1+2^n+3^n....n^n$.

In (51), by writing $2n$ for n, and $-n$ for c, and reducing, we have

$$2[n^{2n}+(n-1)^{2n}+....2^{2n}+1]+1=m'(2n+1)....(53),$$

which is also true only when $2n+1$ is a prime number.

Subtract $2n+1$ from both members of (53), and divide by $2n+1^x$, we have

$$\{[n^{2n}-1]+[n-1)^{2n}-1]+....[2^{2n}-1]\}\div(2n+1)=m''.$$

That is, if each of the quantities in the [ ] is exactly divisible by $2n+1$, then $2n+1$ is a prime number. This may be considered a generalization of Fermat's Theorem.

SOME MODERN METHODS AND PRINCIPLES OF GEOMETRY.*

By PROFESSOR HEINRICH MASCHKE, Ph. D.

It might be said of the most important parts of recent geometry that one conception dominates everywhere: that is the conception of the *group*. Suppose we are given a set of operations of any kind, which I call S_1, S_2, S_3, S_4, ... , finite or infinite in number, a set of operations which are defined by some law. Take now one of the operations, say S_i, apply it first, and after that has been done apply in succession another of the operations, S_k. If now it is so that the combined operation S_iS_k, which is obtained by applying first S_i and afterwards S_k, is again an operation *in the original set;* and if this is so for any two operations of the set, then the set forms a *group*. Let me give you an example. Think of a sphere with center fixed, and define a set of operations by all the possible rotations of the sphere about its center. That is an infinite number of operations. These operations, I say, form a group. Revolve the sphere first about a certain diameter through a certain angle. This is one of the operations of our set. After that has been done, take another axis and revolve the sphere about this second axis through a certain angle. Then it can be proved that the combined effect of these two rotations is equivalent to a single rotation about a certain axis and through a certain angle. The effect produced by two operations of the set applied in succession is the same as the effect of another operation contained in the set. Therefore, all these rotations form a group. The number of operations in this group is infinite.

Suppose now we have a triangle with sides of two, three, and four feet in length. Whether we make an investigation about this triangle here in this room in Ryerson Laboratory, or over in Cobb Hall, say, the result is the same. This means that in geometry our investigations are independent of the location of our figures in space. In other words, if I make a certain investigation of a certain triangle and then move that triangle to some other place in space, I do not change anything of the character of the theorem. Now instead of saying that we will move our figure from one place to another, I will rather say that we move the whole of space by that same amount which will bring this figure into coincidence with the other figure; and so then the following statement will be clear: our geometrical theorems are not changed when we submit the whole of space to a certain motion. The truth of our geometrical theorems is independent of the motion of space. If we consider all the possible motions of the whole of space, then these motions form a group, because the application of two motions in succession is equivalent to one single motion. Every motion can be considered as a transformation in the following sense: Suppose we take a point, and fix it by some means, say by its coördinates x, y, z; then by any motion of the space the point

*Read at the fifteenth educational conference of the academies and high schools affiliating or co-operating with the University of Chicago. With some modifications, the first part of this paper appeared in *The School Review*, January, 1902.

(x, y, z) goes into another point (say x', y', z'); and so every point of space is *transformed* into some other point. What we consider is this transformation, this connection between the points in the old position and the new position. Now, whenever the notion of a group comes in there is always the question of what remains invariant under such a group. If we subject the space to all possible motions, the most important invariant is the distance between two points. Take any two points, A and B; however you may move your space by translation, or rotation, or whatever you like, the distance between A and B rémains always the same: it is an *invariant*. Also the angle between any two lines is invariant under this group of all possible motions in space. Of course these are not the only invariants. Indeed, every geometrical property—the theorem that the three perpendiculars at the middle points of the three sides of a triangle meet in a point, and all similar theorems—is independent of the accidental location of the triangle in space; all these theorems have an invariant character.

Let us go a step further. Take some scalene triangle, ABC, and consider the symmetrical triangle $A'B'C'$—all sides and angles equal respectively, but lying in the opposite direction. It is possible to make them lie one on the other by a certain motion. Take the line of symmetry, and revolve the plane of the first triangle about this line; then this triangle will cover the other one. But such a motion is not possible if you allow only motion in the plane. Let us say the triangle $A'B'C'$ is obtained from ABC by a *reflection* on their line of symmetry. In space, take a certain plane and reflect our figures on this plane. An irregular tetrahedron goes by such a reflection into another precisely equal to the first; but it is not possible by any *motion* in space to bring the two tetrahedrons into coincidence with each other. It is like the difference between the right and left hands. It would be possible to bring them together by mere motion if we could go into a space of four dimensions,* but it is not possible in space of three dimensions; just as in the case of the two triangles, where it is not possible to bring them into coincidence by motion in a plane, but only by motion in space of three dimensions.

But now I say in our geometrical investigations it does not make any difference whether we consider a certain figure or a figure which is deduced from the first one by such a reflection.

Let us consider all possible reflections in space on all possible planes. The question is, do they form a group? The answer is, no, because one reflection on one plane changes a given tetrahedron into a symmetrical tetrahedron, and any other reflection on a second plane changes the second tetrahedron into its symmetrical tetrahedron, which is equal and equally directed to the first, so that by two successive reflections we do not get again a reflection, but something which is equivalent to a motion. If, however, we join to all possible motions of space all possible reflections, this totality again forms a group, because no matter how you combine any motions and reflections, you always get either a motion or a reflection: that is to say, you get again an operation of the set. What is invar-

*As to the space of four dimensions, see the explanations given at the end of the paper.

iant under this group? The distance between any two points, the angle between any two lines, and in the third place, every elementary geomotrical theorem.

Again let us go a step further. Suppose we investigate a triangle with sides respectively two, three, and four feet in length. A teacher in Paris does not say *feet*, but twenty, thirty, forty centimeters—a different size; but the theorems which he deduces from his triangle are the same as the theorems which we deduce. In other words, for our elementary geometrical theorems the size is immaterial. We allow then an expansion or reduction in size, everything remaining similar, of course. To fix the ideas let us define such an expansion or reduction in this way: Take a fixed point, and join it to all points in space by lines called radii vectores, and change every radius vector, without changing the angles, in the ratio $l:n$; the effect will be the expansion or reduction of the whole of space in size. Now let us join to all operations of our group containing all possible motions and reflections all these expansions and reductions; the combined operations again form a group, and this group has been called by Klein the *principal group* of geometry. Our geometrical theorems then remain true under this principal group: that is to say, they remain true if we apply any one of the operations of this principal group—any motion and reflection, or any expansion or reduction in size.

If we ask about invariants, we see at once that under this group distance is not invariant. But the ratio of two distances is invariant; it remains, of course, invariant for every motion and every reflection, and also for every expansion or reduction. The angle between two lines is also an invariant under the principal group. With this conception of the principal group we might give the following definition of the subject-matter of elementary geometry. We might say it is the establishment and deduction of geometrical properties which remain unchanged under this principal group.

Let us now extend this group by joining other operations. We then come right into the midst of modern geometry. Take any plane figure in space, on the board, for instance, and now take a point not in the plane of the board, and join this point to all the points of your figure: let the point be your eye, say, and let the straight lines be the lines on which you look upon the different points. If now you take a plane and place that plane in any position between the point and the board, there results what is called a projection of the figure on the board on this new plane. Let A be a point in the plane of the board, and O your center of projection, and let the corresponding point in the second plane be A', the point of intersection of the plane with OA. Thus every point A goes into a definite point A'. How does this figure in the second plane differ from the figure in the first plane? Is the distance between two points preserved? Certainly not. Is the ratio of the distances of two points preserved? Certainly not in general. If you have the points A and B, and C in the middle, and project from the point O, the point C' will *not* be in the middle of $A'B'$, unless the two planes are parallel. The angles between any two lines are also changed. But there is another thing which remains invariant—the ratio of two ratios. Take the line AB and divide it at C and D. Then

$$\overline{A \qquad C \qquad D \qquad B} \qquad \frac{CA}{CB} : \frac{DA}{DB}$$

is invariant under this projection. This is called the *cross-ratio* or *anharmonic-ratio* between these points. This projection, however, might be considered as a transformation of the plane. Take the second plane and place it on the first plane; then you have on the first plane a certain point A and its corresponding point A', B and its corresponding point B'; whence you have a transformation of the different points on that plane.

A similar transformation is possible in space; only to make that projection we have to take a point outside of space; that is, a point in the fourth dimension somewhere. From that point we project every point of our space into another three-dimensional space, and then bring that second space into coincidence with the first. Then you have the same relation as before—for every point A a new point A'.

Analytically this transformation is much simpler. It can be shown that the coördinates x', y', z' of the new points A' are rational linear functions of the coördinates x, y, z of the old points A. From these formulas it follows at once that all these transformations (they are called *projections* in the plane and *collineations* in space) form a *group*.

Apply to the x', etc., a collineation, and you get x'', etc., in terms of x, y, z, a formula of the same kind. And every formula of that kind gives a collineation. Therefore the totality of all collineations in space form a group. This group contains the principal group, because every motion, every reflection, and every expansion or reduction can always be expressed by a formula of the above kind. This is the *group of projective geometry*.

Here the distance is not any longer invariant, nor is the angle, nor is the ratio between two lines; but the cross-ratio is an invariant, indeed the most important one of this group of projective geometry. The subject matter of projective geometry is then the study of geometrical theorems which remain unchanged under this group.

There are many other possible transformations of space, and each is defined by a certain group. I mention the Cremona transformations, in which the coördinates of the new points are no longer linear, but higher rational functions of the old, and the old of the new. These transformations also form a group, and that group contains all the groups which we had before. Another very general transformation is the transformation which underlies the so-called *analysis-situs*—the investigation of all those geometrical properties which remain unchanged for every continuous *deformation*. By that I mean any deformation which is such that two points which are very near together remain very near together; such a transformation as is made by squeezing a rubber ball in your hand. This transformation is so general, one might think, that by this process we could change any figure into almost any other figure. But by squeezing a ring you can never make a sphere, and conversely, by that process of deforma-

tion you can never get a ring from a sphere. There are also several invariants under this transformation—the most important of which is the so-called *genus*.

There is another principle of modern geometry which I wish to point out in a few words. I have mentioned occasionally the *fourth dimension*. Now the new principle referred to is the free use of any number of dimensions in geometry. Since we are three dimensional beings, it is utterly impossible for us to see in our imagination any space of higher than three dimensions. The study of higher spaces is therefore, and can only be, purely analytical. We might also treat analytic geometry of three dimensions in a purely analytical way, leaving aside all geometrical notions. In this sense analytic geometry of three dimensions is simply the study of functions of three independent variables x, y, z. Likewise, analytic geometry of four dimensions is the study of functions of four independent variables x, y, z, w. But in this study we might borrow the phraseology from analytic geometry of three dimensions. We might talk of a plane, of a line, a point, a three-dimensional space in the space of four dimensions, meaning by these terms certain linear equations or systems of equations in x, y, z, w. One linear equation would represent a three-dimensional space; for instance, $w=0$ would represent the ordinary space of three dimensions. Two linear equations in x, y, z, w would represent a plane; etc. Reasoning by analogy from three-dimensional space will help us then considerably in our analytic study in four dimensions.

In a certain way, however, a direct geometrical insight into spaces of higher dimensions is possible. When we consider our ordinary space as consisting not—as we are accustomed to—of points as elements, but of straight lines, then it becomes at once a space of four dimensions, because a straight line is determined by four independent coördinates. Again, taking other simple figurations as elements of space, for instance, the sphere, the circle, or the general surface of the second order, we might endow our ordinary space with any number of dimensions we please.

In geometry of three dimensions there are only five *regular* bodies: the tetrahedron, the hexahedron, the octahedron, the dodekahedron, and the ikosahedron. If we wish to represent these regular figures of space in the plane, we take a plane and a point outside, and project on the plane the regular hexahedron, for example. In general, several of the projected edges will meet. But that can be easily avoided in the following way: Place the body under consideration on the plane, and take as point of projection a point above the middle point of one of the faces and not far from it, in such a way that the upper face is so projected that it includes all the other faces. Then no two projected edges meet. For instance, Fig. 1 shows the projection of the regular hexahedron.

Let us do the same thing in a higher space. Take the space of four dimensions. It can be shown that in this space there are six regular bodies. Our imagination fails of course to see them, but we can see the projections of these bodies into our space of three dimensions. As the center of projection, we take

a point in the space of four dimensions chosen so that no meeting of the various lines occur.

A body of four dimensions is bounded by what corresponds to faces in the body of three dimensions —*i. e.*, by a certain number of bodies of three dimensions, in such a way that all these different bodies lie in difierent spaces; and every one of these is bounded by planes, every plane by edges, and every edge by vertices.

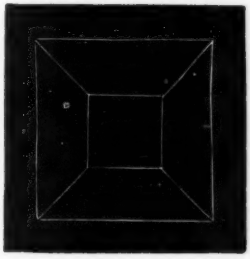


Fig. 1.

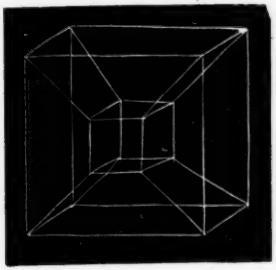


Fig. 2.

In Fig. 2 is given a perspective view of the projection of the so-called 8-cell, one of the regular bodies in four-dimensional space. We observe in the figure eight hexahedrons (counting also the one which includes all the others); these are the projections of the three-dimensional bodies (cells) which bound the four-dimensional body.

In the lecture itself, a set of wire models belonging to the mathematical department of the University of Chicago was shown to illustrate the projections into space of three dimensions of all six regular four-dimensional bodies.

The University of Chicago, October, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

148. Proposed by R. D. BOHANNAN, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\frac{x}{a+\alpha}+\frac{y}{b+\beta}+\frac{z}{c+\gamma}=1$, $\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=1$, $\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1$, show, without solving, that $x+y+z=a+\alpha+b+\beta+c+\gamma$.

Solution by JAMES McMAHON, A. M., Professor of Mathematics, Cornell University, Ithaca, N. Y.

There is probably a misprint in the first equation. It should be

$$\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}=1.$$

The form of the three given equations shows that α, β, γ are the three roots of the equation

$$\frac{x}{a+s}+\frac{y}{b+s}+\frac{z}{c+s}=1,$$

in which s is regarded as the unknown. On clearing of fractions, and arranging in the form of a cubic equation in s, it is seen that the sum of the three roots is $-(a+b+c)+(x+y+z)$.

Hence $\alpha+\beta+\gamma=-(a+b+c)+(x+y+z)$, and $x+y+z=\alpha+a+b+\beta+c+\gamma$.

NOTE. It may be of interest to state that if each letter be squared the result expresses the distance of any point from the origin in terms of ellipsoidal curvilinear coördinates.

156. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

$(z+x)a-(z-x)b=2yz....(1)$; $(x+y)b-(x-y)c=2xz....(2)$; $(y+z)c-(y-z)a=2xy....(3)$. Find the values of x, y, and z by the method of linear simultaneous equations.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x=\frac{1}{2}(b+c)u$, $y=\frac{1}{2}(a+c)v$, $z=\frac{1}{2}(a+b)w$.

$\therefore (a-b)w+(b+c)u=(a+c)vw....(1)$.

$(b-c)u+(a+c)v=(a+b)uw....(2)$.

$(c-a)v+(a+b)w=(b+c)uv....(3)$.

We might eliminate v, w and get an equation of the fifth degree in u. We will, however, proceed as follows: Add (1), (2), (3), then

$$aw(2-u-v)+bu(2-v-w)+cv(2-u-w)=0.$$

This is the case when $u=v=w=0$; or $u=v=w=1$; or $u=0$, $w=v=2$; or $v=0$, $u=w=2$; $w=0$, $u=v=2$.

The first two sets of values satisfy the conditions.

$\therefore x=y=z=0$; $x=\frac{1}{2}(b+c)$, $y=\frac{1}{2}(a+c)$, $z=\frac{1}{2}(a+b)$.

NOTE. This is exercise 31, page 224, Systems of Linear Simultaneous Equations, of Fisher and Schwatt's *Higher Algebra*, and has given teachers of algebra throughout the country considerable trouble. Solving the equations for a, b, and c, we readily find that

$$a=-x+y+z,$$
$$b=x-y+z, \text{ and}$$
$$c=x+y-z.$$

$\therefore x=\frac{1}{2}(b+c)$, $y=\frac{1}{2}(a+c)$, $z=\frac{1}{2}(a+b)$, as one set of values for x, y, and z. EDITOR F.

Also solved by L. C. WALKER.

157. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Solve the equations

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\gamma}+\frac{u}{d+\lambda}=1,\quad \frac{x}{a+\mu}+\frac{y}{b+\mu}+\frac{z}{c+\mu}+\frac{u}{d+\mu}=1,$$

$$\frac{x}{a+\nu}+\frac{y}{b+\nu}+\frac{z}{c+\nu}+\frac{u}{d+\nu}=1,\quad \frac{x}{a+\rho}+\frac{y}{b+\rho}+\frac{z}{c+\rho}+\frac{u}{d+\rho}=1.$$

I. Solution by L. E. DICKSON, A. M., Ph. D., Assistant Professor of Mathematics, University of Chicago, Chicago, Ill.

In view of the given equations, the following equation in t,

$$\frac{x}{a+t}+\frac{y}{b+t}+\frac{z}{c+t}+\frac{u}{d+t}=1 \ldots . (1)$$

has the roots λ, μ, ν, ρ. Setting $f(t)\equiv(t+a)(t+b)(t+c)(t+d)$, we have

$$xf_a(t)+yf_b(t)+zf_c(t)+uf_d(t)=f(t),$$

where $f_a(t)$ denotes the partial derivative of $f(t)$ with respect to a. Hence

$$f(t)-xf_a(t)-yf_b(t)-zf_c(t)-uf_d(t)\equiv(t-\lambda)(t-\mu)(t-\nu)(t-\rho)$$

is an identity in t. To obtain the value of x, set $t=-a$. Then

$$-xf_a(-a)=(-1)^4(a+\lambda)(a+\mu)(a+\nu)(a+\rho),$$

$$x=\frac{(a+\lambda)(a+\mu)(a+\nu)(a+\rho)}{(a-b)(a-c)(a-d)}.$$

The value of any other variable, say y, may be obtained similarly, or by interchanging a with b in the expression for x, as is evident from the symmetry of (1).

This method of solution is equally simple for n such equations in n variables.

II. Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

By Art. 586 of Hall and Knight's *Higher Algebra*, consider the following equation in θ,

$$\frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{z}{c+\theta}+\frac{u}{d+\theta}=1-\frac{(\theta-\lambda)(\theta-\mu)(\theta-\nu)(\theta-\rho)}{(a+\theta)(b+\theta)(c+\theta)(d+\theta)};$$

x, y, z, u being for the present regarded as known quantities.

This equation when cleared of fractions is of the *third* degree in θ, and is

satisfied by the *four* values $\theta=\lambda$, $\theta=\mu$, $\theta=\nu$, $\theta=\rho$, in virtue of the given equations; hence it must be an identity.

To find the value of x, multiply up by $a+\theta$, and then put $a+\theta=0$; thus

$$x=\frac{(a+\lambda)(a+\mu)(a+\nu)(a+\rho)}{(a-b)(a-c)(a-d)}.$$

By symmetry, we have

$$y=\frac{(b+\lambda)(b+\mu)(b+\nu)(b+\rho)}{(b-c)(b-d)(b-a)},$$

$$z=\frac{(c+\lambda)(c+\mu)(c+\nu)(c+\rho)}{(c-d)(c-a)(c-b)},$$

$$\text{and } u=\frac{(d+\lambda)(d+\mu)(d+\nu)(d+\rho)}{(d-a)(d-b)(d-c)}.$$

Similarly solved by *G. B. M. ZERR*.

GEOMETRY.

185. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Given the tangential equations to two conics S, S', find the tangential co-ordinates of the join of the poles of two given parallel lines with respect to S. Deduce the tangential equation of the center of S, and find that of the intersection of S and S'.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $bc-f^2=A$, $ca-g^2=B$, $ab-h^2=C$, $gh-af=F$, $hf-bg=G$, $fg-ch=H$.

Then $S=A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu$.

Similarly, $S'=A'\lambda^2+B'\mu^2+C'\nu^2+2F'\mu\nu+2G'\nu\lambda+2H'\lambda\nu$.

Let $\lambda\alpha+\mu\beta+\nu\gamma$ and $\lambda\alpha+\mu\beta+\nu\gamma+m$ be the two given parallel lines; p, q, t and p', q', t' their poles with respect to S. Then for the first line

$$ap+hq+gt=\lambda,\ hp+bq+ft=\mu,\ gp+fq+ct=\nu.$$

Solving these equations for p, q, t,

$$p=(A\lambda+H\mu+G\nu)/\triangle,\ q=(H\lambda+B\mu+F\nu)/\triangle,$$
$$t=(G\lambda+F\mu+C\nu)/\triangle,\ \text{where } \triangle=abc+2fgh-af^2-bg^2-ch^2.$$

For the second line,

$$ap'+hq'+gt'+m=\lambda,\ hp'+bq'+ft'+m=\mu,\ gp'+fq'+ct'+m=\nu.$$

$\therefore\ p'=[A\lambda+H\mu+G\nu-m(A+H+G)]/\triangle.$

$q'=[H\lambda+B\mu+F\nu-m(H+B+F)]/\triangle.$

$t'=[G\lambda+F\mu+C\nu-m(G+F+C)]/\triangle.$

(p, q, t), (p', q', t') are the tangential co-ordinates of the join of the poles.

Let A', B', C' be the angles of the triangle of reference. The center is the pole of the line at infinity $\alpha\sin A'+\beta\sin B'+\gamma\sin C'=0$. The tangential co-ordinates of the center are obtained by substituting $\sin A'$, $\sin B'$, $\sin C'$ for λ, μ, ν in p, q, t and are

$$S_1=(A\sin A'+H\sin B'+G\sin C')/\triangle,$$
$$S_2=(H\sin A'+B\sin B'+F\sin C')/\triangle,$$
$$S_3=(G\sin A'+F\sin B'+C\sin C')/\triangle.$$

$\therefore$ The tangential equation of the center is $\lambda S_1+\mu S_2+\nu S_3=0$.

Write $a+ka'$ for a, $b+kb'$ for b, $c+kc'$ for c, $f+kf'$ for f, $g+kg'$ for g, $h+kh'$ for h in $A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0$.

Then the tangential equation of the four points of intersection of S and S' is $S+k\Phi+k^2S'=0$ where k is undetermined, and

$$\Phi=(bc'+b'c-2ff')\lambda^2+(ca'+c'a-2gg')\mu^2+(ab'+a'b-2hh')\nu^2$$
$$+2(gh'+g'h-af'-a'f)\mu\nu+2(hf'+h'f-bg'-b'g)\pi\lambda$$
$$+2(fg'+f'g-ch'-c'h)\lambda\mu.$$

The condition for equal roots for k is $\Phi^2=4SS'$, which is the equation of the four points of intersection.

186. Proposed by J. R. HITT; Professor of Mathematics, Coronal Institute, San Marcos, Texas.

If two sides of a triangle and its in-circle be given in position, tne envelope of its circumcircle is a circle (*Mannheim*). [From Casey's *Sequel to Euclid*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let vertex A be origin, sides b, c the axes. Then $x^2+2xy\cos A+y^2-bx-cy=0$ is the equation to the circumcircle. Let this equation be written

$$D-bx-cy=0....(1).$$

Since the sides b, c and the inscribed circle are fixed in position, the tangents from A to the in-circle are constant.

$\therefore\ b+c-a=$ a constant $=m....(2)$.

$a=\sqrt{(b^2+c^2-2bc\cos A)}$. This in (2) gives after reduction,

$$m^2+2bc(1+\cos A)-2m(b+c)=0....(3).$$

c from (1) in (3) gives

$$2b^2x(1+\cos A)+2b[my-mx-D(1+\cos A)]+2Dm-m^2y=0.$$

The condition for equal roots of b is

$$2x(1+\cos A)(2Dm-m^2y)=[my-mx-D(1+\cos A)]^2$$

$$\text{or } [D(1+\cos A)-m(x+y)]^2=2m^2xy(1-\cos A)=4m^2xy\sin^2\tfrac{1}{2}A.$$

$$\therefore D(1+\cos A)-m(x+y)\pm 2m\sqrt{(xy)}\sin\tfrac{1}{2}A=0.$$

$$\therefore x^2+2xy\cos A+y^2-\frac{m}{2\cos^2\frac{1}{2}A}[x+y\pm 2\sqrt{(xy)}\sin\tfrac{1}{2}A]=0.$$

$$\therefore x^2+2xy+y^2-4xy\sin^2\tfrac{1}{2}A-\frac{m}{2\cos^2\frac{1}{2}A}[x+y\pm 2\sqrt{(xy)}\sin\tfrac{1}{2}A]=0.$$

$$\therefore [x\pm 2\sqrt{(xy)}\sin\tfrac{1}{2}A+y][x\mp 2\sqrt{(xy)}\sin\tfrac{1}{2}A+y-\frac{m}{2\cos^2\frac{1}{2}A}]=0.$$

$$\therefore x\mp 2\sqrt{(xy)}\sin\tfrac{1}{2}A+y-\frac{m}{2\cos^2\frac{1}{2}A}=0,$$

$$\text{or } x^2+2xy\cos A+y^2+[m-4m(x+y)\cos^2\tfrac{1}{2}A]/4\cos^4\tfrac{1}{2}A=0.$$

This is the circle.

CALCULUS.

144. Proposed by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the volume of the sphere, $x^2+y^2+z^2=2az$, (a) within the paraboloid $z=Ax^2+By^2$; (b) within the cone $z^2=Ax^2+By^2$.

Solution by the PROPOSER.

$x^2+y^2+z^2=2az....(1)$, $z=Ax^2+By^2....(2)$, $z^2=Ax^2+By^2....(3)$.

From (1), $z=a\pm\sqrt{\{a^2-x^2-y^2\}}=a\pm\sqrt{\{a^2-r^2\}}$.

From $v=\int\int zrdrd\theta$ we get

$$v=4\int_0^{\frac{1}{2}\pi}\int_0^R\sqrt{(a^2-r^2)}d\theta rdr=\tfrac{4}{3}\int_0^{\frac{1}{2}\pi}]a^3-(a^2-r^2)^{\frac{3}{2}}][d\theta.$$

(a). From (1) and (2),

$$x^2+y^2+(Ax^2+By^2)^2=2a(Ax^2+By^2).$$

$$r^2+r^4(A\cos^2\theta+B\sin^2\theta)^2=2ar^2(A\cos^2\theta+B\sin^2\theta).$$

$$\therefore r^2 = \frac{2a(A\cos^2\theta + B\sin^2\theta) - 1}{(A\cos^2\theta + B\sin^2\theta)^2} = R^2.$$

$$\therefore v = \frac{4a^3}{3}\int_0^{\frac{1}{2}\pi}\left[1 - \left(\frac{A\cos^2\theta + B\sin^2\theta - 1}{A\cos^2\theta + B\sin^2\theta}\right)^3\right]d\theta$$

$$= \frac{4a^3}{3}\int_0^{\frac{1}{2}\pi}\left[\frac{3}{A\cos^2\theta + B\sin^2\theta} - \frac{3}{(A\cos^2\theta + B\sin^2\theta)^2} + \frac{1}{(A\cos^2\theta + B\sin^2\theta)^3}\right]d\theta$$

$$= \frac{2\pi a^3}{\sqrt{AB}}\left[1 - \frac{A+B}{2AB} + \frac{A^2+B^2}{8A^2B^2}\right] + \frac{\pi a^3}{6\sqrt{\{A^3B^3\}}}.$$

(*b*). From (1) and (3), $R = \dfrac{2a\sqrt{\{A\cos^2\theta + B\sin^2\theta\}}}{1 + A\cos^2\theta + B\sin^2\theta} = \dfrac{2aC}{D}$.

$$\therefore v = \frac{4a^3}{3}\int_0^{\frac{1}{2}\pi}\left[1 - \left(\frac{C^2-1}{D}\right)^3\right]d\theta = \frac{8a^3}{3}\int_0^{\frac{1}{2}\pi}\left(\frac{3}{D} - \frac{6}{D^2} + \frac{4}{D^3}\right)d\theta$$

$$= \frac{2\pi a^3}{\sqrt{\{(A+1)(B+1)\}}}\left[\left(\frac{A}{A+1}\right)^2 + \left(\frac{B}{B+1}\right)^2\right] + \frac{4\pi a^3}{3\sqrt{\{(A+1)^3(B+1)^3\}}}.$$

145. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the surface bounding the volume required in problem 102.

Solution by the PROPOSER.

The surface is composed of a portion of the surface of the cone, and a portion of the surface of the paraboloid.

The equation to the cone is $x^2 + y^2 = c^2z^2$.

The equation to the paraboloid is $y^2 + z^2 = 4a(a+x)$.

$S = S_c + S_p$.

$$S_c = \frac{2}{c}\int_{x_2}^{x_1}\int_0^{y_1}\sqrt{\{1+c^2\}}\,dxdy, \text{ \{see 102 for limits\}}$$

$$= \frac{2}{c}\int_{x_2}^{x_1}\sqrt{\{4ac^2(a+x) - x^2\}}\,dx.$$

Let $x = 2ac^2 - 2ac\sqrt{\{1+c^2\}}\cos\theta$.

$$\therefore S_c = 8a^2c(1+c^2)\int_0^{\pi}\sin^2\theta\,d\theta = 4\pi a^2c(1+c^2).$$

$$S_p = 4\int_{x_2}^{x_1}\int_0^{y_1}\sqrt{\frac{a(2a+x)}{4a(a+x) - y^2}}\,dxdy = 4\int_{x_2}^{x_1}\sqrt{\{a(2a+x)\}}\sin^{-1}\sqrt{\frac{4ac^2(a+x) - x^2}{4a(1+c^2)(a+x)}}\,dx$$

$$= \frac{4}{3a}\int_{x_2}^{x_1} \frac{x(2a^2+ax)^{\frac{3}{2}}}{(a+x)\sqrt{\{4ac^2(a+x)-x^2\}}}\,dx$$

$$= \frac{4a}{3}\int_{x_2}^{x_1} \frac{x\sqrt{\{2a^2+ax\}}dx}{(a+x)\sqrt{\{4ac^2(a+x)-x^2\}}} + \tfrac{4}{3}\int_{x_2}^{x_1} \frac{x\sqrt{\{2a^2+ax\}}dx}{\sqrt{\{4ac^2(a+x)-x^2\}}}.$$

Let $x=2ac^2+2ac\sqrt{\{1+c^2\}}\cos 2\theta$, $1+c^2+c\sqrt{\{1+c^2\}}=b^2$, $2c\sqrt{\{1+c^2\}}/b^2=e^2$, $4c\sqrt{\{1+c^2\}}/\{2b^2-1\}=d$.

$$\therefore S_p = \frac{16a^2b\sqrt{2}(b^2-1)}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$- \frac{16a^2b\sqrt{2}(2c\sqrt{\{1+c^2\}}+b^2e^2-e^2)}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$+ \frac{32a^2bce^2\sqrt{\{2(1+c^2)\}}}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$+ \tfrac{16}{3}a^2b\sqrt{2}(b^2-1)\int_0^{\frac{1}{2}\pi}\sqrt{\{1-e^2\sin^2\theta\}}d\theta$$

$$- \tfrac{32}{3}a^2bc\sqrt{\{2(1+c^2)\}}\int_0^{\frac{1}{2}\pi}\sqrt{\{1-e^2\sin^2\theta\}}\sin^2\theta d\theta.$$

$$\therefore S_p = A\Pi(e, -d, \tfrac{1}{2}\pi) + \frac{B}{d}\{F(e, \tfrac{1}{2}\pi) - \Pi(e, -d, \tfrac{1}{2}\pi)\}$$

$$+ \frac{C}{d^2e^2}\{e^2\Pi(e, -d, \tfrac{1}{2}\pi) + dE(e, \tfrac{1}{2}\pi) - (d+e^2)F(e, \tfrac{1}{2}\pi)\} + DE(e, \tfrac{1}{2}\pi)$$

$$+ \frac{E}{3e^2}\{1-2e^2)E(e, \tfrac{1}{2}\pi) - (1-e^2)F(e, \tfrac{1}{2}\pi)\}.$$

$$\therefore S = 4\pi a^2c(1+c^2) + (A - \frac{B}{d} + \frac{C}{d^2})\Pi(e, -d, \tfrac{1}{2}\pi) + \Big(\frac{C}{de^2} + D$$

$$+ \frac{E(1-2e^2)}{3e^2}\Big)E(e, \tfrac{1}{2}\pi) + \Big(\frac{B}{d} - \frac{C(d+e^2)}{d^2e^2} - \frac{E(1-e^2)}{3e^2}\Big)F(e, \tfrac{1}{2}\pi).$$

MECHANICS.

139. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A homogeneous sphere, radius $r=50$ inches, makes $m=30$ revolutions around an axis every second. The mass begins to disappear from the surface into space at a rate exactly sufficient to cause the diameter to decrease uniformly at the rate of $(1/n)$th$=1/1000$th of

a linear inch per second. At what rate per second is the angular velocity of the sphere changing the instant the diameter becomes $p=10$ inches? What is the diameter of the sphere when the rate of disappearance of matter is midway between minimum and maximum? When is the angular velocity a maximum? How does this maximum angular velocity compare with the original angular velocity? What is the diameter of the sphere when the paracentric force is (1) a maximum, and (2) a minimum?

No solution of this problem has been received.

140. Proposed by J. F. LAWRENCE, A. B., St. Louis, Mo.

A long row of particles, each mass m, is placed on a smooth horizontal table. Each is connected with the two adjacent ones by similar light elastic strings of natural length l. They receive small longitudinal disturbances such that each of them proceeds to perform a harmonic oscillation. Prove that there will be two waves of vibration in opposite directions with the same velocity, viz, $l'\sqrt{\frac{E}{ml}}\frac{q}{\pi}\sin\frac{\pi}{q}$, when l' is the average distance between two successive particles, q the number of intervals between two particles in the same phase, and E the modulus of elasticity. [*Mathematical Tripos*, 1873.]

Solution by G. B. M ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $D=d/dt$, and the equation of motion is $y_{k+1}-2y_k+y_{k-1}=\frac{D^2}{c^2}y_k$ where $c^2=E/(ml)$. To solve this equation of differences, treat D as constant and put $y_k=Cx^k$ where C and x are two constants. Substituting and reducing, $x-2+1/x=(D/c)^2$.

$$\therefore \sqrt{x}-1/\sqrt{x}=\pm D/c. \quad \sqrt{x}+1/\sqrt{x}=\pm\sqrt{\{4+(D/c)^2\}}=\pm 2\sqrt{1+\left(\frac{D}{2c}\right)^2}$$

$$\therefore \sqrt{x}=\sqrt{1+\left(\frac{D}{2c}\right)^2}-\frac{D}{2c}=E.$$

$$\therefore y_k=E^{2k}f(t)+E^{-2k}F(t).$$

Let us express $f(t)$ and $F(t)$ in a series whose general term is $A\cos(2c\sin\theta t+\omega)$.

The operator E under the radical contains only even powers of D and we can write $-(2c\sin\theta)^2$ for D^2.

$$\therefore E\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega-\theta).$$

$$\therefore E^{2k}\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega-2k\theta).$$

$$E^{-2k}\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega+2k\theta).$$

$$\therefore y_k=\Sigma A\cos(2c\sin\theta t+\omega-2k\theta)+\Sigma B\cos(2c\sin\theta t+\omega+2k\theta).$$

If we substitute in any one term of the first series $k+1$ for k and $t+T$ for t, where $T=\frac{\theta}{c\sin\theta}$, the term is unaltered..

$\therefore$ Any one term of the first series represents a wave which travels the space between one particle and the next in time T. In the same way the corresponding term of the second series represents a wave which travels in the opposite direction with the same velocity.

Now v=velocity=$l'/T=l'c\sin\theta/\theta$; but $\theta=\pi/q$ and $c=\sqrt{\frac{E}{ml}}$.

$$\therefore v=l'\sqrt{\frac{E}{ml}}\frac{q}{\pi}\sin\frac{\pi}{q}.$$

141. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. England.

A simple pendulum hangs from a bicycle moving in a straight line. What deflection is produced by putting on the brake so as to exert on the machine a force equal to the nth of its weight?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let v=velocity of retardation, W=weight.

Then $v=\frac{Wg}{nW}t=\frac{g}{n}t$. Let $t=1$. $\therefore v=g/n$.

If θ=angle of deflection of the pendulum, and l=its length, then $v^2=g^2/n^2=2gl(1-\cos\theta)$.

$$\therefore \cos\theta=\frac{2ln^2-g}{2ln^2} \text{ or } \theta=\cos^{-1}\left(\frac{2ln^2-g}{2ln^2}\right).$$

142. Proposed by GEORGE R. DEAN, B. Sc., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo,

An infinite mass of liquid is bounded by the plane zx, on which are small corrugations given by $y=\phi(x)$. The velocity of the liquid at an infinite distance from the plane is parallel to x and equal to V. Prove that the velocity potential is $V_x+\frac{V}{\pi}\int_{-\infty}^{\infty}\frac{(x-\lambda)\phi(\lambda)d\lambda}{y^2+(x-\lambda)^2}$. [Bassett's *Hydrodynamics.*]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let f=velocity potential.

$$\text{Then } \frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}=\frac{d^2f}{dx^2}\left(1-\frac{1}{[\varphi'(x)]^2}\right)=0.$$

$$\therefore \frac{df}{dx}=\text{constant}=v \text{ when } y=\infty. \quad \therefore f-C=v\int dx=v\varphi_1(x).$$

When $y=0$, $f=\varphi_1(x)v$; when $y=\infty$, $\varphi(x)=x$.

$\therefore C=vx$. $\therefore f=vx+v\varphi_1(x)$.

$$\text{By Fourier's series, } \varphi_1(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(\lambda)d\lambda\int_0^{\infty}\sin\beta(x-\lambda)d\beta.$$

We must build up a value of $\varphi_1(x)$ satisfying the last expression.

Take $e^{-\beta y}\sin\beta x$ and $e^{-\beta y}\cos\beta x$ and multiply the first by $\cos\beta\lambda$, the second by $\sin\beta\lambda$. Subtract these and we get $e^{-\beta y}\sin\beta(x-\lambda)$.

$$\therefore\ \varphi_1(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{-\beta y}\varphi(\lambda)\sin\beta(x-\lambda)d\beta d\lambda.$$

$$\text{Now}\ \int_0^\infty e^{-ax}\sin mx dx=\frac{m}{a^2+m^2}.$$

$$\therefore\ \int_0^\infty e^{-\beta y}\sin\beta(x-\lambda)d\beta=\frac{x-\lambda}{y^2+(x-\lambda)^2}.$$

$$\therefore\ \varphi_1(x)=\frac{1}{\pi}\int_{-\infty}^\infty\frac{(x-\lambda)\,\varphi(\lambda)d\lambda}{y^2+(x-\lambda)^2}.$$

$$\therefore f=vx+\frac{v}{\pi}\int_{-\infty}^\infty\frac{(x-\lambda)\,\varphi(\lambda)d\lambda}{y^2+(x-\lambda)^2}.$$

143. Proposed by **W. J. GREENSTREET, M. A.**, Editor of The Mathematical Gazette, Stroud, England.

Beads are fastened at equal intervals on a string placed over a smooth fixed pulley. If the original position of the string is one of symmetry, find the velocity at any moment, the pressure on the pulley, and the velocity with which the string leaves the pulley.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

In this solution the weights of the pulley and string are neglected.

Let $2l$=length of string, $2n+1$=number of beads each mass m.

Acceleration$=f=\dfrac{m[(n+x)-(n-x+1)]g}{m(2n+1)}=\dfrac{2x-1}{2n+1}g$, where x can have any value from 1 to $(n-1)$. l/n=distance between two beads.

$$\text{Now}\ l/n=\tfrac{1}{2}ft_1^2=\frac{(2x-1)gt_1^2}{2(2n+1)}.$$

$$\therefore\ t_1=\sqrt{\frac{2l(2n+1)}{gn(2x-1)}},\ v=ft_1=\sqrt{\frac{2gl\,(2x-1)}{n(2n+1)}}.$$

Let t be any time between the passing of the $(x+1)$th and $(x+2)$th particle over the pulley, then the velocity, v_1, at this time, is

$$v_1=\sum_{x=1}^{x=x}\sqrt{\frac{2gl\,(2x-1)}{n(2n+1)}}+\sqrt{\frac{2gl\,(2x+1)}{n(2n+1)}}t=\sqrt{\frac{2gl}{n(2n+1)}}[1+\sqrt{3}+\sqrt{5}+\sqrt{7}+\ldots.$$

$$+\sqrt{(2x-1)}+\sqrt{(2x+1)}t].$$

The velocity with which the string leaves the pulley

$$=\Sigma_{x=1}^{x=n-1}\sqrt{\frac{2gl\,(2x-1)}{n(2n+1)}}=\sqrt{\frac{2gl}{n(2n+1)}}[1+\sqrt{3}+\sqrt{5}+\ldots.+\sqrt{(2n-3)}].$$

Pressure on pulley at time $t=\dfrac{4m^2(n+x+1)(n-x)}{m(2n+1)}=\dfrac{4m(n+x+1)(n-x)}{2n+1}$

DIOPHANTINE ANALYSIS.

92. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the sides of integral right triangles when the difference of the legs is given.

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Let a be the difference in the legs, and x and $x+a$=legs. Then $2x^2+2ax+a^2=\square=$(say) $[px-a]^2=p^2x^2-2apx+a^2$. Whence, $x=\dfrac{2a[p+1]}{p^2-2}$. Take $p=2$, $x=3a$, and the sides are $3a$, $4a$, $5a$. Then in the formula $\dfrac{2[r+s]}{r+2s}$, we have $r/s=\frac{2}{1}$, then $p=\frac{3}{2}$, and $x=20a$, and the sides $20a$, $21a$, and $29a$, and so on *ad infinitum*.

Remark on Problem 94 by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

In his solution of this question, Professor Zerr gives up his general demonstration a little prematurely. It is true that "For integral values of m, m^2+m+1 is not a square," but fractional values of m lead to integral values of a, b, and c. The value of m which makes the expression a square is $\dfrac{2p+1}{p^2-1}$ in which p may be any number except one. Take $p=2$, and $m=\frac{5}{3}$. Then $a=\frac{40}{9}$, $b=\frac{24}{9}$, and $c=\frac{15}{9}$. Any common multiple of a, b, and c makes the square root of $\sqrt{[a^2+b^2+c^2]}$ a square and as the denominator is a square, makes abc a square, as well as these particular values. So a, b, and c may be taken=40, 24, and 15, respectively. Then $abc\sqrt{[a^2+b^2+c^2]}=14400=49$, a square number. However, the question does not call for integral values, and I had solved the question as follows from the point at which Professor Zerr leaves it. Substituting the value of $m=\dfrac{2p+1}{p^2-1}$ in the values of a, b, and c, reducing to common denominator, $[p^2-1]^2$, we have $a=p[2p^2+5p+2]$, $b=p[p+2][p^2-1]$, and $c=2p^3=p^2-2p-1$, in which p may be any number except one. Hence, there is an indefinite number of rational triangles whose area is a square.

96. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

(*a*) Find the least three integral numbers such that the difference of every two of them shall be a square number; (*b*) find the least three square numbers such that the difference of every two af them shall be a square number.

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

(a). Take a=one of the numbers, any then if we take x^2+a and y^2+a =the other numbers, and two of the conditions are met, and we have only to make x^2-y^2 (the difference between the other two) a square. But x^2-y^2 is the expression for one of the sides of a right angled triangle. Hence we may take $x=p^2+q^2$ and $y=p^2-q^2$, or $2pq$ and the three numbers will be $[p^2+q^2]^2+a$ and $[2pq]^2+a$ or $[p^2-q^2]^2+a$ and a, in which a, p, and q may be any numbers. To obtain the least three take $a=1$, $p=2$, and $q=1$, and the numbers are 1, 10 or 17, and 26, the least being 1, 10 and 26.

(b). The solution of the second part of this problem is on page 113, April number. The problem was incorrectly numbered 92. ED.

97. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Find a general expression for the radius of the sphere which, dropped in (or partly in) a right cone full of water, will displace the most water; the radius of the sphere, and the width, height and slant height of the cone to be rational integral numbers.

Solution by the PROPOSER.

Let ABC be a section of the cone passing through the axes, AB being the diameter of the cone, CD the axis, BC the slant height, and EB and EF radii of the sphere. Let $BD=a$, $BC=b$, and $CD=c$. Let $BE=BF=x$ and $DE=y$; then $BD=\sqrt{[x^2-y^2]}$; then $\frac{\pi[x+y]^3}{6}+\frac{\pi[x^2-y^2][x+y]}{2}$=contents of sphere within the cone which must be a maximum. Omitting constants, this reduces to $2x^3+3x^2y-y^3$=maximum....[1].

But $BC:BD::CE:FE$; or $b:a::c-y:x$, and $bx=ac-ay$....[2].

Differentiating, and reducing, $dy=-bdx/a$. Differentiating [1], substituting the value of dy and reducing, we have $2ax[x+y]-b[x^2-y^2]=0$. Dividing by $x+y$, we have $2ax-b[x-y]=0$....[3].

$x+y=0$ gives $x=-y$. Substituting this value in [1], the expression becomes zero. Hence, this value of x does not answer the conditions of the question. Substituting the value of y in [3] as found in [2], and reducing, we have

$$x=\frac{abc}{[b-a][b+2a]}.$$

As a, b, and c are sides of a right angled triangle, take $b=p^2+q^2$, $a=p^2-q^2$, and $c=2pq$. Then

$$x=\frac{p[p^2+q^2][p^2-q^2]}{q[3p^2-q^2]}.$$

For integrals, $x=p[p^2+q^2][p^2-q^2]$, $2a=2q[p^2-q^2][3p^2-q^2]$,

$$b=q[p^2+q^2][3p^2-q^2],\quad c=2pq^2[3p^2-q^2],$$

in which p and q may be any integral numbers, p being greater than q.

E. g. Take $p=2$ and $q=1$. Then $x=30$, $2a=66$, $b=55$, and $c=44$.

Of course, we can take $a=2pq$, and $c=p^2-q^2$; but the resulting expression is not so simple and yet when reduced to integrals gives the same value for x, but cone thus obtained has different dimensions.

Also solved by G. B. M. Zerr, who gets slant height$=605m^2$, altitude$=484m^2$, radius of sphere $=330m^2$, and radius of cone$=363m^2$, where m is any integer.

98. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

(a) Find the least three integral numbers such that if to the square of each the product of the other two be added, the three sums shall all be squares.

(b) Find the two least integral numbers such that not only each of them, but also their sum and their difference, when increased by unity, shall all be square numbers.

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

(a) Let x^2, m^2x^2 and nx^2 represent the required numbers. Then we have $1+mn=\square$....(1); $m^2+n=\square$....(2), and $m+n^2=\square$....(3). Take $n=\frac{1}{4}-m$ ....(4), and (2) becomes a square, as (3) does, when we transpose (4) and get $m=\frac{1}{4}-n$. Substitute the value of n in (1) and we have

$$1+\frac{m}{4}-m^2=\square=(\text{say})\ [1-pm]^2.$$

Reducing, we have $m=\dfrac{8p+1}{4[p^2+1]}$ and $n=\frac{1}{4}-m=\dfrac{p[p-8]}{4[p^2+1]}$.

Take $x=4[p^2+1]$, $mx=8p+1$, and $nx=p[p-8]$, in which p may be any number greater than 8. Take $p=9$ and we have $x=328$, $mx=73$, and $nx=9$.

(b) Let x^2 and y^2 be the numbers. Then $x^2+y^2+1=\square=(\text{say})(m+1)^2$ and $x^2-y^2+1=\square=(\text{say})(m-1)^2$.

By adding and reducing, we get $x^2=m^2$ and $x=m$.

By substituting and reducing, we find $y^2=2m$. Take $m=2n^2$ and we have $x=2n^2$ and $y=2n$. Take $n=1$, and $x=2$, and $y=2$, and the numbers are 4 and 4; take $n=2$ and the numbers are 16 and 64.

Also solved by *G. B. M. ZERR*, and *L. C. WALKER*.

AVERAGE AND PROBABILITY.

Remark on Problem 109, by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

Let $r=a\sin^2\theta$. Then the last integral in Professor Walker's solution is transformed as follows:

$$\frac{8}{9a}\int_0^a[3a^3\tan^{-1}(\frac{r}{a-r})-(3a^2+2ar-8r^2)\sqrt{(ar-r^2)}]dr,$$

$$=\frac{16a^3}{9}\int_0^{\frac{1}{2}\pi}(3\theta-3\sin\theta\cos\theta-2\sin^3\theta\cos\theta+8\sin^5\theta\cos\theta)\sin\theta\cos\theta d\theta,$$

$$= \frac{16a^3}{9}\int_0^{\frac{1}{2}\pi}(3\theta\sin\theta\cos\theta - 3\sin^2\theta + \sin^4\theta + 10\sin^6\theta - 8\sin^8\theta)d\theta,$$

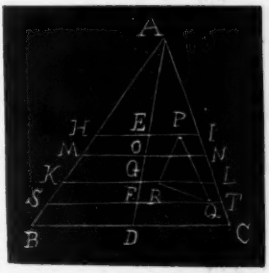
$$= \frac{16a^3}{9}\left(\tfrac{3}{8}\pi - \tfrac{3}{4}\pi + \tfrac{3}{16}\pi + \tfrac{5}{16}\pi - \tfrac{3}{32}\pi\right) = \frac{16a^3}{9}\times\frac{9\pi}{32} = \tfrac{1}{2}\pi a^3.$$

117. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A straight line is drawn at random parallel to the base of a given triangle. Three random points are then taken, one on each side of the random line and one anywhere in the triangle. Find the average area of the triangle formed by the three random points.

Solution by the PROPOSER.

Let ABC be the given triangle, MN the random line parallel to the base BC. Take P above, Q below MN, and R anywhere. Through P, Q, R draw HI, ST, KL parallel, respectively, to MN. Let $BC=2a$, $BD=DC=a$, $AD=h$, $\angle ADC=\beta$, $AE=u$, $AF=v$, $AG=w$, $EP=x$, $GR=z$, $FQ=y$, $AO=r$.



Then $EI=\frac{au}{h}=x'$, $GL=\frac{aw}{h}=z'$, $FT=\frac{av}{h}=y'$,

$GJ=y-\frac{(y-x)(v-w)}{v-u}=t$.

Area $PQR=\frac{1}{2}(t-z)(v-u)\sin\beta=A$, $t>z$; area $PQR=\frac{1}{2}(z-t)(v-u)\sin\beta=A_1$, $t<z$. The limits of r are 0 and h; of v, r and h; of u, 0 and r; of w, u and v; of x, $-x'$ and x'; of y, $-y'$ and y'; of z, $-z'$ and t, and t and z'.

$$\therefore \Delta = \frac{\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\int_u^v\left[\int_{-z'}^{t}A\,dz+\int_{t}^{z'}A_1\,dz\right]drdvdydudxdw}{\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\int_u^v\int_{-z'}^{z'}drdvdydudxdwdz}$$

$$= \frac{15}{a^3h^4}\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\int_u^v\left[\int_{-z'}^{t}A\,dz+\int_{t}^{z'}A_1\,dz\right]drdvdydudxdw$$

$$= \frac{15\sin\beta}{2a^3h^6}\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\int_u^v\left[h^2t^2+a^2w^2\right](v-u)drdvdydudxdw$$

$$= \frac{5\sin\beta}{2a^3h^6}\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\left[a^2(u^2+uv+v^2)+h^2(x^2+xy+y^2)\right]$$

$$\times(v-u)^2drdvdydudx$$

$$=\frac{5\sin\beta}{3a^2h^7}\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r(4a^2u^3+3a^2u^2v+3a^2uv^2+3h^2uy^2)(v-u)^2\,drdvdydu$$

$$=\frac{5\sin\beta}{36a^2h^7}\int_0^h\int_r^h\int_{-y'}^{y'}[3a^2r^2v^2-12a^2rv^3+18a^2v^4-12a^2r^3v+8a^2r^4$$

$$+3h^2y^2(6v^2-8rv+3r^2)]r^2\,drdvdy$$

$$=\frac{5a\sin\beta}{9h^8}\int_0^h\int_r^h(3r^4v^3-10r^3v^4+12r^2v^5-6r^5v^2+4r^6v)\,drdv$$

$$=\frac{5a\sin\beta}{36h^8}\int_0^h(3r^2h^4-8rh^5+8h^6-8r^3h^3+8r^4h^2-3r^6)r^2\,dr$$

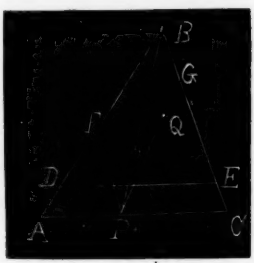
$$=\tfrac{1\,3}{1\,2\,6}ah\sin\beta=\tfrac{1\,3}{1\,2\,6}\text{ (area of triangle).}$$

115. Proposed by F. P. MATZ. Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Find the mean distance between two points taken at random in an equilateral triangle.

I. Solution by L. C. WALKER. A. M., Graduate Student, Leland Stanford University, Cal.

Let ABC be any plane triangle whose sides opposite the angles A, B, C are respectively a, b, c. Let M be the required mean distance, and M_1 be the mean distance between the two points when one of the points is confined to one of the sides of the triangle, as AC. Let P be any point in the side AC, and let Q be any point in the surface of the triangle whose area$=\triangle$. Draw DE parallel to AC, and draw PF, PB, and PQG. Put $BD=u$, $AP=x$, $PQ=y$, $PF=z$, $PG=v$, $\angle BPC=\phi$, and $\angle FPC$ or $\angle GPC=\theta$. Then



$$v=\frac{c\sin A\sin(\phi+C)}{\sin\phi\sin(\theta+C)},\quad z=\frac{c\sin A\sin(\phi-A)}{\sin\phi\sin(\theta-A)},$$

$$x=\frac{c\sin(\phi-A)}{\sin\phi},\text{ and }dx=\frac{c\sin A}{\sin^2\phi}d\phi.$$

$$\text{Hence } M=\frac{\int_0^c M_1\cdot\frac{u}{c}\cdot\frac{u^2}{c^2}\triangle\cdot DE\,du}{\int_0^c\frac{u^2}{c^2}\triangle\cdot DE\,du}=\tfrac{4}{5}M_1$$

$$=\frac{4}{5b\triangle}\int_0^b\left[\int_0^\phi\int_0^v y^2\,dyd\theta+\int_\phi^\pi\int_0^z y^2\,dyd\theta\right]dx$$

$$= \frac{8c^3\sin^3 A}{15b^2}\int_A^{\pi-C}\Big[\int_0^{\phi}\frac{\sin^3(\phi+C)}{\sin^3\phi}\cdot\frac{d\theta}{\sin^3(\theta+C)}$$

$$+\int_{\phi}^{\pi}\frac{\sin^3(\phi-A)}{\sin^3\phi}\cdot\frac{d\theta}{\sin^3(\theta-A)}\Big]\frac{d\phi}{\sin^2\phi} = \frac{4c^3\sin^3 A}{15b^2}\int_A^{\pi-C}\Big[\frac{\cos C\sin^3(\phi+C)}{\sin^2 C\sin^3\phi}$$

$$+\frac{\cos A\sin^3(\phi-A)}{\sin^2 A\sin^3\phi} - \frac{\sin(\phi+C)\cos(\phi+C)}{\sin^3\phi} + \frac{\sin(\phi-A)\cos(\phi-A)}{\sin^3\phi}$$

$$+\frac{\sin^3(\phi+C)}{\sin^3\phi}\log\cot\tfrac{1}{2}C + \frac{\sin^3(\phi-A)}{\sin^3\phi}\log\cot\tfrac{1}{2}A - \frac{\sin^3(\phi+C)}{\sin^3\phi}\log\cot\tfrac{1}{2}(\phi+C)$$

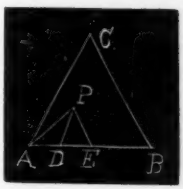
$$+\frac{\sin^3(\phi-A)}{\sin^3\phi}\log\cot\tfrac{1}{2}(\phi-A)\Big]\frac{d\phi}{\sin^2\phi} = \tfrac{1}{15}\Big[\Big(\frac{a^2}{b}+\frac{b^2}{a}\Big)(\cos C+\sin^2 C\log\cot\tfrac{1}{2}C)$$

$$+\Big(\frac{a^2}{c}+\frac{c^2}{a}\Big)(\cos B+\sin^2 B\log\cot\tfrac{1}{2}B)+\Big(\frac{b^2}{c}+\frac{c^2}{b}\Big)(\cos A+\sin^2 A\log\cot\tfrac{1}{2}A)\Big].$$

Corollary. If $a=b=c$, then $M=\frac{a}{20}(4+3\log 3)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $\triangle$ be the required average, $\triangle_0$ the average distance from an angle to a point taken at random within the triangle. Let $AP=r$, $AB=a$, $\angle PAB=\theta$. The limits of r are 0 and $\frac{1}{2}a\sqrt{3}\operatorname{cosec}(\frac{2}{3}\pi-\theta)=x$; of θ, 0 and $\frac{1}{3}\pi$.



$$\therefore\ \triangle_0 = \frac{4}{a^2\sqrt{3}}\int_0^{\frac{1}{3}\pi}\int_0^{x} r^2\,d\theta\,dr = \tfrac{1}{2}a\int_0^{\frac{1}{3}\pi}\operatorname{cosec}^3(\tfrac{2}{3}\pi-\theta)\,d\theta$$

$$= a(\tfrac{1}{3}+\tfrac{1}{4}\log 3).$$

From P draw PD, PE parallel to the sides and let the area of ABC=unity. The average distance from P to any point in PDE is $u(\frac{1}{3}+\frac{1}{4}\log 3)$ where $u=DE$. Since either of the vertices of PDE can be taken,

$$\triangle = 6\int_0^{a}\int_0^{a-x} u(\tfrac{1}{3}+\tfrac{1}{4}\log 3)u^2\,du\,dv = \tfrac{3}{5}a(\tfrac{1}{3}+\tfrac{1}{4}\log 3).$$

$$\therefore\ \triangle = \tfrac{3}{5}\triangle_0.$$

119. Proposed by F. L. SAWYER, Mitchell, Ontario, Canada.

Two players throw three dice, the object being to throw at one cast a tres, a deux, and an ace. They continue throwing in succession until one of the players wins. What advantage has the first player?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

The chance of winning on the first throw is $\frac{1}{2\,1\,6}$; the chance of first losing and second throw winning is $\frac{215}{(216)^2}$; the chance of first and second losing and third winning is $\frac{(215)^2}{(216)^3}$; the chance of the first, second and third losing and fourth winning is $\frac{(215)^3}{(216)^4}$; etc.

The chance of the first player winning is

$$\tfrac{1}{2\,1\,6}\left[1+(\tfrac{2\,1\,5}{2\,1\,6})^2+(\tfrac{2\,1\,5}{2\,1\,6})^4+(\tfrac{2\,1\,5}{2\,1\,6})^6+\ldots.\right]=\tfrac{2\,1\,6}{4\,3\,1}.$$

The chance of the second player winning is

$$\tfrac{1}{2\,1\,6}\left[\tfrac{2\,1\,5}{2\,1\,6}+(\tfrac{2\,1\,5}{2\,1\,6})^3+(\tfrac{2\,1\,5}{2\,1\,6})^5+(\tfrac{2\,1\,5}{2\,1\,6})^7+\ldots.\right]=\tfrac{2\,1\,5}{4\,3\,1}.$$

$\therefore$ The first player's chance : the second player's chance $=216 : 215$.

Also solved by *J. SCHEFFER*, who gets 36:35. This result is wrong. Professor Zerr's solution is correct.

Lon C. Walker sent in a solution of problem 115.

MISCELLANEOUS.

112. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

(a) Find the area enclosed by four circles, of which two touch the x-axis, and two the y-axis at the origin.

(b) Required the area enclosed by four parabolas, of which two touch the x-axis, and two the y-axis, at the origin.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

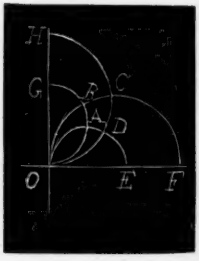
The area $ABCD$ is the area required.

(a) Let $r=a\cos\theta$....(1), $r=b\cos\theta$....(2), $r=c\sin\theta$....(3), $r=d\sin\theta$....(4), be the equations to the circles. $a>b$, $c>d$.

Area common to (1) and (3) $=A$; area common to (1) and (4) $=B$; area common to (2) and (3) $=C$; area common to (2) and (4) $=D$. Let $\tan^{-1}(a/c)=\theta'$.

$$A=\tfrac{1}{2}a^2\int_{\theta'}^{\frac{1}{2}\pi}\cos^2\theta d\theta+\tfrac{1}{2}c^2\int_0^{\theta'}\sin^2\theta d\theta=\tfrac{1}{4}a^2\cot^{-1}(a/c)$$

$$+\tfrac{1}{4}c^2\tan^{-1}(a/c)-\tfrac{1}{4}ac.$$



$$\therefore\ B=\tfrac{1}{4}a^2\cot^{-1}(a/d)+\tfrac{1}{4}d^2\tan^{-1}(a/d)-\tfrac{1}{4}ad.$$

$$C=\tfrac{1}{4}b^2\cot^{-1}(b/c)+\tfrac{1}{4}c^2\tan^{-1}(b/c)-\tfrac{1}{4}bc.$$

$$D=\tfrac{1}{4}b^2\cot^{-1}(b/d)+\tfrac{1}{4}d^2\tan^{-1}(b/d)-\tfrac{1}{4}bd.$$

Area $ABCD=A-C+B-D=\tfrac{1}{4}a^2[\cot^{-1}(a/c)$

$$+\cot^{-1}(a/d)]-\tfrac{1}{4}b^2[\cot^{-1}(b/c)+\cot^{-1}(b/d)]+\tfrac{1}{4}c^2[\tan^{-1}(a/c)-\tan^{-1}(b/c)]$$

$$+\tfrac{1}{4}d^2[\tan^{-1}(a/d)-\tan^{-1}(b/d)]-\tfrac{1}{4}a(c+d)+\tfrac{1}{4}b(c+d)$$

$$=\tfrac{1}{4}a^2\cot^{-1}\left(\frac{a^2-cd}{ac+ad}\right)-\tfrac{1}{4}b^2\cot^{-1}\left(\frac{b^2-cd}{bc+bd}\right)+\tfrac{1}{4}c^2\tan^{-1}\left(\frac{ac-bc}{ab+c^2}\right)$$

$$+\tfrac{1}{4}d^2\tan^{-1}\left(\frac{ad-bd}{ab+d^2}\right)-\tfrac{1}{4}(a-b)(c+d).$$

(b) Let $r=4a\cos\theta/\sin^2\theta$....(1), $r=4b\cos\theta/\sin^2\theta$....(2), $r=4c\sin\theta/\cos^2\theta$....(3), $r=4d\sin\theta/\cos^2\theta$....(4), be the equations to the parabolas; $a>b$, $c>d$.

Let $\tan^{-1}\sqrt{(a/c)}=\theta'$.

$$A=8a^2\int_{\theta'}^{\frac{1}{2}\pi}\frac{\cos^2\theta}{\sin^4\theta}d\theta+8c^2\int_0^{\theta'}\frac{\sin^2\theta}{\cos^4\theta}d\theta=\tfrac{16}{3}ac.$$

Similarly, $B=\frac{16}{3}ad$, $C=\frac{32}{3}bc$, $D=\frac{16}{3}bd$.

Area $ABCD=A+B-C-D=\frac{16}{3}(a-b)(c+d)$.

113. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

Deduce the Sylvestrian Reciprocant from $x^4+y^4=4x^2y^2$.

Solution by the PROPOSER.

The given equation may be written $(x^2-y^2)^2=2x^2y^2$.

$\therefore\ x^2-y^2=xy\sqrt{2}$....($a$).

Put $(y/x)=w$; then from (a), by dividing by xy, etc., we get $w-w^{-1}=-\sqrt{2}$:

$\therefore\ w=\frac{1}{2}(-\sqrt{2}\pm\sqrt{3}),=m$....($b$).

$\therefore\ y=mx$, and $dy/dx=m$....(c).

Eliminating m from (c) by a second differentiation, we have $d^2y/dx^2=0$ ....(d).

Adopting Professor Sylvester's notation for reciprocants, viz: $dy/dx=t$, $d^2y/dx^2=a\ 2!$, $d^2y/dx^3=b\ 3!$, etc., we obtain from (d) the *first* pure Sylvestrian Reciprocant. All reciprocants *not* containing $t=dy/dx$ are *pure*; all others are *mixed*. The first two pure reciprocants are a and $4ac-5b^2$, and the first two mixed ones are $(1+t^2)b-2a^2t$ and $bt-a^2$. See our paper on Reciprocants published in the MONTHLY of November, 1894.

Also solved by G. B. M. ZERR.

114. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

When the sun's declination was 15° N. his altitude was found to be 20°, and after an hour's interval his altitude was found to be 31°. Required, the latitude of the place of observation.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $a=20°$=altitude, $a'=31°$=altitude, φ=latitude, $\delta=15°$=declination, $\theta=15°$=sun's angular path for one hour, h=hour angle.

$$\therefore \cos h=\frac{\sin a-\sin\varphi\sin\delta}{\cos\varphi\cos\delta},\ \cos[h-\theta]=\frac{\sin a'-\sin\varphi\sin\delta}{\cos\varphi\cos\delta}.$$

Eliminating h, we get

$$[\sin a-\sin\varphi\sin\delta]\cos\theta+\{\cos^2\varphi\cos^2\delta-[\sin a-\sin\varphi\sin\delta]^2\}^{\frac{1}{2}}\sin\theta=\sin a'-\sin\varphi\sin\delta.$$

But $\theta=\delta$. $\therefore \sin^2\varphi\sin^2\delta[2+\cos\delta]-2\sin\varphi\sin\delta[1-\cos\delta][\sin a+\sin a']$

$$=\cos^2\delta\sin^2\delta+2\sin a\sin a'\cos\delta-\sin^2 a-\sin^2 a'.$$

$$\therefore .067055\sin^2\varphi-.015126\sin\varphi=.020553;\ \sin^2\varphi-.2256\sin\varphi=.3065.$$

$$\sin\varphi=.6778 \text{ or } -.4522;\ \varphi=42°\ 40'\ 30''.$$

115. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

Determine geometrically where to stand so as to be able to throw a stone over a tree with the minimum velocity.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The velocity of projection is the same as a body would acquire in falling from the directrix of the parabolic path to the point of projection.

$\therefore$ The velocity will be a minimum when the directrix is the least distance above the top of the tree. This is the case when you stand at the base of the tree, then the directrix passes just above the tree.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

164. Proposed by JOSEPH V. COLLINS, Ph. D., Professor of Mathematics, State Normal School, Stevens Point, Wis.

Three women, the first with ten eggs, the second with thirty, and the third with fifty, went to market. They each got the same for their eggs, and all returned with the same money. What did they get?

ALGEBRA.

168. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If n, $n+2$, $n+6$, $n+8$, $n+12$ are all primes, find the form of n.

169. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $x^2+y+z=a$....(1),
$x+y^2+z=b$....(2),
$x+y+z^2=c$....(3).

GEOMETRY.

191. Proposed by J. V. McADAMS, St. Louis, Mo.

Trisect any angle by means of the hypocycloid.

192. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

Of all triangles with a common base and inscribed in the same circle, the isosceles is the maximum and has the maximum perimeter. Prove geometrically.

CALCULUS.

158. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

It is required to cut a hole a inches square, for a crank shaft, through the center of a grindstone b inches thick at the outer edge, c inches thick at the center, and d inches in diameter. How many cubic inches will have to be cut out?

159. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve $\dfrac{d^2u}{dx^2}=\dfrac{1}{m}\left(\dfrac{du}{dt}\right)$.

MECHANICS.

148. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are *electrical images* of each other in respect to a given sphere may be constant, and that their centers of gravity should lie on the surface of the sphere.

149. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

From two points in the same horizontal line hangs a light inextensible string, on which are threaded two beads of equal mass. The beads start from rest in the position in which the terminal portions of the string are vertical and move symmetrically towards each other in the vertical plane. Find the path of each bead, and the tension of the string at any point in the path.

DIOPHANTINE ANALYSIS.

108. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

The determinant $\triangle$ of the special group-matrix for the Quaternion Group equals $\sigma_1\sigma_2\sigma_3\sigma_4(x_1-x_2)^4$, where $\sigma_1=x_1+x_2+2x_3+2x_5+2x_7$, $\sigma_2=x_1+x_2+2x_3-2x_5-2x_7$, $\sigma_3=x_1+x_2-2x_3+2x_5-2x_7$, $\sigma_4=x_1+x_2-2x_3-2x_5+2x_7$. Find the order of the linear group for the $GF[p^n]$, namely, the number of sets x_1, x_2, x_3, x_5, x_7, for which $\triangle$ is not equal to 0.

AVERAGE AND PROBABILITY.

133. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

A circle of unknown radius is drawn with its center at the vertex of a given parabola, and has its greatest area when its circumference passes through the focus of the parabola. Required the average area common to the circle and parabola.

134. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

An ellipse, semi-axes a, b, is placed on a square, side c. Find the chance that center of ellipse is on the square.

MISCELLANEOUS.

130. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

From a cloud of angular elevation $\phi=45°$, a streak of lightning darted to the earth. The temperature of the atmosphere was $t=80°$, and the percentage of humidity $p=90$. After $m=3$ seconds, the report of the stroke at the earth was heard. How far away from the observer did the streak of lightning (1) start, and (2) strike the earth?

NOTES.

With this number of the MONTHLY begins a new arrangement for its future management which, we hope, will be as thoroughly appreciated by our readers as it is gratifying to us, and which, we are confident, will bring to it increased power and usefulness.

While in Chicago the first of September, we called on Dr. Dickson and urged him to join us in the editorship of the MONTHLY. Not seeing his way clear at the time, he withheld his answer until he could consider the matter. After some meditation, he decided affirmatively.

We feel that we are especially fortunate in securing the coöperation of so valuable a man as Dr. Dickson. He stands in the very front of the younger generation of mathematicians. Though not yet thirty years old, his contributions to mathematics in the way of original and important articles published in the various mathematical and scientific journals of the world is truly marvelous—

the most of his contributions being on the Theory of Groups. He is also the author of a *College Algebra* published by John Wiley & Sons; an *Introduction to Algebraic Equations* now being published by the same publishers, and a work on *Linear Groups*, published by B. G. Teubner, of Leipzig, Germany.

He is also on the editorial staff of the *Transactions* of the American Mathematical Society.

In 1896, Dr. Dickson took his doctor's degree at the University of Chicago; he then spent a year in study in Europe, one semester at Leipzig, and one semester at Paris. Returning to America, he became Instructor and later Assistant Professor of Mathematics in the University of California; then for one year Associate Professor of Mathematics in the University of Texas; and in 1900 he was called to the University of Chicago as Assistant Professor of Mathematics.

While Dr. Dickson has devoted the most of his time to research work and thereby materially aiding in enlarging the domain of mathematics, yet he is conscious of the importance of devoting time to the improvement of elementary texts, and thus placing the fundamentals of mathematics on a firm foundation. To this end he prepared his works on Algebra.

He thus brings to his work in the co-editorship of the MONTHLY a thorough and comprehensive knowledge of elementary as well as of higher mathematics. He will have full charge of the editing of all papers intended for publication in the MONTHLY. Therefore, we ask our contributors to send all articles for publication directly to him. His address is Box 142, Faculty Exchange, The University of Chicago.

All problems and solutions as well as all business communications pertaining to the MONTHLY should be sent, as heretofore, to B. F. Finkel, Springfield, Mo.

We hope that all our readers and contributors will coöperate with the editors in every way possible to make the MONTHLY a greater power for good than ever. B. F. F.

The hundredth anniversary of the birth of Abel, the eminent Norwegian mathematician, was celebrated during September at Christiania. Representative scientists from many countries were present. Among those upon whom honorary degrees were conferred were Simon Newcomb and J. Willard Gibbs. L. E. D.

Prof. J. M. Colaw, formerly associate editor of the MONTHLY, is coöperation with Prof. J. K. Ellwood, is now busily engaged in preparing for publication a text-book of Algebra. We are glad to note that the series of Arithmetics prepared by these gentlemen is meeting with wonderful success. B. F. F.

The recent address by John Purser, President of Section A of the British Association was an "Historical Sketch of the Irish School of Mathematics and Physics." It dealt at length on the life of Sir W. R. Hamilton. L. E. D.

Mr. J. W. Miller, Ph. D. (Columbia) has been appointed Instructor in Mathematics and Astronomy in Lehigh University. L. E. D.

BOOKS AND PERIODICALS.

Plane and Spherical Trigonometry. An elementary text-book. By Charles H. Ashton, A. M., Instructor in Mathematics in Harvard University, and Walter R. Marsh, A. B., Head Master Pingry School, Elizabeth, N. J. 8vo. Cloth, x+157 pages. New York: Charles Scribner's Sons.

Two distinguishing features of this text may be noted. The first is the systematic distinction (except in the first two chapters on the functions of acute angles and the solution of right triangles) between positive and negative directions of straight lines. The second is the proof of the addition theorem by the method of projection (as well as the usual proof). These features show that the text is thoroughly up to date. Among the minor points of pedagogical excellence is the *introduction* of the cosecant, secant, and cotangent as the reciprocals of the sine, cosine, and tangent, respectively. The book is well printed and the page attractive. L. E. D.

Elements of the Theory of the Newtonian Potential Functions. By B. O. Peirce, Ph. D., Hollis Professor of Mathematics and Natural Philosophy in Harvard University. Third Edition, Revised and Enlarged. 8vo. Cloth, xiii+490 pages. Price, $1.50. Boston and Chicago: Ginn & Co.

The chief difference between this and the first edition is in the way of additions. The first edition contained 178 pages, this contains 490 pages. In the present edition, a full discussion is given of electrostatics and electrokinematics. In this edition, 40 pages are devoted to a treatment of Electromagnetism, against none in the first edition. Current Induction is also more fully treated. Under each chapter is given a list of problems the solution of which will fix in the mind the principles under discussion. The work ends with a list of 387 problems some of which will tax the powers of the mathematician.

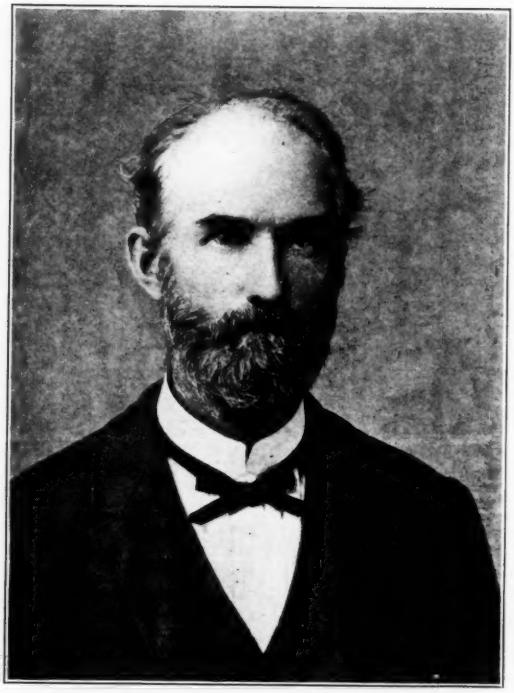
This scholarly book is a valuable contribution to mathematical Physics. It is very clear and easily understood by students who have a good working knowledge of mechanics and the calculus. B. F. F.

Elements of Physics. By Fernando Sanford, Professor in Leland Stanford Junior University. 8vo. Cloth, xxxi+426 pages. Price, $1.20. New York: Henry Holt & Co.

In this text-book, Mechanics is based on the energy concept from the beginning. In the treatment of properties of bodies, the gaseous state is treated before the liquid and the solid states. The author justifies this by the relative simplicity of the gaseous state of aggregation as compared with the liquid or solid states. Considerable stress is laid on the kinetic theory of gases. The greatest departure from established usage in elementary text-books is made in the subject of Optics. In this book Optics is treated geometrically from the beginning. This book contains only such experiments as are essential to the successful teaching of the subject. It is well written, but in the absence of a fairly well equipped laboratory, it is deficient in illustrations of important pieces of apparatus. B. F. F.

A Short Course in Plane and Spherical Trigonometry. By Edwin S. Crawley, Ph. D., Thomas A. Scott Professor of Mathematics in the University of Pennsylvania. 8vo. Cloth, 116+xxvii pages. Price, with tables, $1.00. Published by the author.

This is, in part, an abridgment of the author's "Elements of Plane and Spherical Trigonometry." It is most admirably adapted to the needs of the better class of High Schools and Normal Schools, and may be used to advantage in most colleges. B. F. F.



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A MATRIX DEFINED BY THE QUATERNION GROUP.

By DR. L. E. DICKSON.

1. The Quaternion Group was chosen by Weber* to illustrate his exposition of a portion of Frobenius's theory of group-matrices and group-determinants.† In his treatment of the illustrative example, Weber gives no clue to the reader how his results were obtained originally and makes the verification depend upon two laborious compositions of matrices of order eight. The example may, however, be treated very simply by a method which emphasizes certain remarkable properties enjoyed by group-matrices.

A second purpose of this paper is to furnish an instructive example of the theory of group-matrices as generalized for an arbitrary field (Körper, realm of rationality) by the writer.‡ The canonical forms are essentially different in the cases of a field having modulus 2 and a field not having modulus 2. The methods used are elementary and the paper is entirely independent of those cited. Incidentally, it illustrates a method of factoring important determinants.

2. The Quaternion Group may be defined by the multiplication-table§

*Weber, *Algebra*, edition of 1899, vol. 2, pp. 216-218; pp. 125-128.

†For an elementary, but complete, exposition of Frobenius's theory, see the writer's article in the *Annals of Mathematics*, October, 1902.

‡Dickson, *Transactions American Mathematical Society*, vol. 3 (1902), pp. 285-301; further developments in *University of Chicago Decennial Publications*, vol. IX, pp. 35-51.

§The arrangement of the multipliers in the first column is such that the products equal to the identity element x_1 all appear in the main diagonal.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_2	x_2	x_1	x_4	x_3	x_6	x_5	x_8	x_7
x_4	x_4	x_3	x_1	x_2	x_7	x_8	x_6	x_5
x_3	x_3	x_4	x_2	x_1	x_8	x_7	x_5	x_6
x_6	x_6	x_5	x_8	x_7	x_1	x_2	x_3	x_4
x_5	x_5	x_6	x_7	x_8	x_2	x_1	x_4	x_3
x_8	x_8	x_7	x_5	x_6	x_4	x_3	x_1	x_2
x_7	x_7	x_8	x_6	x_5	x_3	x_4	x_2	x_1

The body of this table is a matrix M of order eight called the *group-matrix* of the Quaternion Group. Consider a transformation T on eight variables $\xi_1, \ldots, \xi_8$ whose coefficients form the matrix M and are elements of a given field F.

3. Suppose first that the field F does not have modulus 2, so that division by 2 is possible in the field. An inspection of matrix M leads to certain linear functions which transformation T multiplies by constants:

$$\eta_1=\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_7+\xi_8,\quad \eta_3=\xi_1+\xi_2-\xi_3-\xi_4+\xi_5+\xi_6-\xi_7-\xi_8,$$
$$\eta_2=\xi_1+\xi_2+\xi_3+\xi_4-\xi_5-\xi_6-\xi_7-\xi_8,\quad \eta_4=\xi_1+\xi_2-\xi_3-\xi_4-\xi_5-\xi_6+\xi_7+\xi_8.$$

Indeed, T replaces them by the respective functions

$$(1)\qquad \eta_1'=\sigma_1\eta_1,\ \eta_2'=\sigma_2\eta_2,\ \eta_3'=\sigma_3\eta_3,\ \eta_4'=\sigma_4\eta_4,$$

where the following abbreviations have been used:

$$\sigma_1=x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8,\ \sigma_3=x_1+x_2-x_3-x_4+x_5+x_6-x_7-x_8,$$
$$\sigma_2=x_1+x_2+x_3+x_4-x_5-x_6-x_7-x_8,\ \sigma_4=x_1+x_2-x_3-x_4-x_5-x_6+x_7+x_8.$$

From the definitions of $\eta_1, \eta_2, \eta_3, \eta_4$, we get

$$\xi_1+\xi_2=\tfrac{1}{4}(\eta_1+\eta_2+\eta_3+\eta_4),\ \xi_5+\xi_6=\tfrac{1}{4}(\eta_1-\eta_2+\eta_3-\eta_4),$$
$$\xi_3+\xi_4=\tfrac{1}{4}(\eta_1+\eta_2-\eta_3-\eta_4),\ \xi_7+\xi_8=\tfrac{1}{4}(\eta_1-\eta_2-\eta_3+\eta_4).$$

Hence $\eta_1, \eta_2, \eta_3, \eta_4$ may be introduced as new independent variables in place of four of the original variables. Also, if we set

$$s_1=\xi_1-\xi_2,\ s_2=\xi_3-\xi_4,\ s_3=\xi_5-\xi_6,\ s_4=\xi_7-\xi_8,$$

the functions $\eta_1, \eta_2, \eta_3, \eta_4, s_1, s_2, s_3, s_4$ are evidently linearly independent functions of $\xi_1, \ldots, \xi_8$ and may be introduced as new variables in place of the latter. Now T replaces s_1, s_2, s_3, s_4 by the respective functions